

\mathbf{Z}_n -QUASIALGEBRAS

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ABSTRACT Recently we have reformulated the octonions as quasissociative algebras (quasialgebras) living in a symmetric monoidal category. In this note we provide further examples of quasialgebras, namely ones where the nonassociativity is induced by a \mathbf{Z}_n -grading and a nontrivial 3-cocycle.

1 INTRODUCTION

Standard methods for dealing with the nonassociativity of the octonions involve much weaker conditions than associativity (as alternative algebras), with the resulting problem that most usual ideas from linear algebra do not go through for them. In [1] we have introduced a solution to this problem based on modern ideas from category theory and quantum group theory [3]. In this approach we work with algebras which *are* associative but only up to a certain ‘rebracketing isomorphism’. A powerful result from category theory[2] then says that one may make all categorical constructions exactly as for associative algebras and afterwards insert the brackets in a consistent manner. We call such objects *quasialgebras*. The paper [1] particularly studied examples of quasialgebras $k_F G$ which are obtained by twisting from group algebras kG , a class that we show includes the octonions and higher Cayley algebras. The rebracketing isomorphism is controlled by a 3-cocycle which in these cases is a coboundary $\phi = \partial F$. However, the theory is much more general than this and can include other much more novel nonassociative objects. In this note we provide examples with ϕ not a coboundary and hence definitely going beyond the form $k_F G$. Our examples are related to 3-cocycles on the group \mathbf{Z}_n and our main results include a complete classification of the possibilities for these for low n .

2 QUASIALGEBRAS

We recall briefly the general categorical setting which we use[3][4]. A monoidal category \mathcal{C} means objects V, W, Z , etc equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, and a collection

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of functorial isomorphisms

$$\Phi_{V,W,Z} : (V \otimes W) \otimes Z \rightarrow V \otimes (W \otimes Z)$$

called the rebracketting *associator* between any three objects. It is required to obey the *pentagon identity*

$$\begin{array}{ccc}
 & (V \otimes W) \otimes (Z \otimes U) & \\
 \Phi \nearrow & & \searrow \Phi \\
 ((V \otimes W) \otimes Z) \otimes U & & V \otimes (W \otimes (Z \otimes U)) \\
 \Phi \otimes \text{id} \searrow & & \nearrow \text{id} \otimes \Phi \\
 (V \otimes (W \otimes Z)) \otimes U & \xrightarrow{\Phi} & V \otimes ((W \otimes Z) \otimes U)
 \end{array}$$

which says that the two ways to reverse the brackettings as shown coincide. Mac Lane's coherence theorem then says that all other routes between two bracketted tensor products also coincide. In effect, this means that one may generalise constructions in linear algebra exactly as if \otimes were strictly associative, dropping brackets. Afterwards one may add brackets, for example putting all brackets accumulating to the left, and then insert applications of Φ as needed for the desired compositions to make sense; all different ways to do this will yield the same net result.

So working in such a category is no harder than usual associative linear algebra. For example, an algebra A in such a category means

$$\bullet \circ (\bullet \otimes \text{id}) = \bullet \circ (\text{id} \otimes \bullet) \circ \Phi_{A,A,A}$$

for the product \bullet , where Φ is inserted for the bracketting to make sense. So, recognising the octonions as such a *quasiassociative algebra* (or *quasialgebra* for short) makes them as good as associative in the precise sense explained above.

LEMMA 1 *Let G be a group and $\phi : G \times G \times G \rightarrow k$ invertible and a cocycle:*

$$\phi(y, z, w)\phi(x, yz, w)\phi(x, y, z) = \phi(x, y, zw)\phi(xy, z, w), \quad \phi(x, e, y) = 1$$

$x, y, z, w \in G$. Then the category of G -graded vector spaces is monoidal with

$$\Phi_{V,W,Z}((v \otimes w) \otimes z) = v \otimes (w \otimes z)\phi(|v|, |w|, |z|)$$

on elements of degree $|v|, |w|, |z| \in G$.

An algebra in this category is called a *G -graded quasialgebra* and is by definition a G -graded vector space with product respecting the grading and obeying

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)\phi(|a|, |b|, |c|), \quad \forall a, b, c \in A$$

of homogeneous degree. There is also a notion of quasicommutativity

$$a \cdot b = b \cdot a\mathcal{R}(|a|, |b|)$$

where a quasibicharacter \mathcal{R} with respect to ϕ defines a braiding or ‘generalised transposition’ in the category. The octonions are both quasiassociative and quasicommutative in the category of $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ -graded spaces with

$$\phi(\vec{x}, \vec{y}, \vec{z}) = (-1)^{(\vec{x} \times \vec{y}) \cdot \vec{z}}, \quad \mathcal{R}(\vec{x}, \vec{y}) = \begin{cases} 1 & \text{if } \vec{x} = 0 \text{ or } \vec{y} = 0 \text{ or } \vec{x} = \vec{y} \\ -1 & \text{else} \end{cases}$$

where we use a vector notation for the grading. Explicitly the octonion product in the graded basis is[1]

$$e_{\vec{x}} \cdot e_{\vec{y}} = e_{\vec{x}+\vec{y}}(-1)^{\sum_{i \leq j} x_i y_j + y_1 x_2 x_3 + x_1 y_2 x_3 + x_1 x_2 y_3}.$$

3 \mathbf{Z}_n -GRADED CASE

In this section we classify the possible cocycles on $G = \mathbf{Z}_n$ for low n and give examples of quasialgebras of this type. We use an additive notation throughout.

LEMMA 2 *Let ϕ be a cocycle on \mathbf{Z}_n with n even. The element $x = \frac{n}{2}$ obeys $\phi(x, x, x) = \omega$ with $\omega^2 = 1$. Moreover, $\phi(x, x, y) = \omega\phi(x, x, x + y)$ for all y .*

Proof Using the cocycle condition and ϕ trivial when any element is the group identity, we have $\phi(x, x, x)^2 = \frac{\phi(2x, x, x)\phi(2x, x, x)}{\phi(x, 2x, x)} = 1$. The other result is also immediate. \diamond

COROLLARY 3 *A \mathbf{Z}_2 -graded quasialgebra is either an associative superalgebra or quasiassociative with $\phi(x, y, z) = (-1)^{xyz}$, $\forall x, y, z \in \mathbf{Z}_2$. The latter is not a coboundary.*

Proof We have $\phi(x, 0, y) = \phi(0, x, y) = \phi(x, y, 0) = 1, \forall x, y \in \mathbf{Z}_2$. By the last lemma for $x = 1$ we have only two choices: $\phi(1, 1, 1) = 1$ or $\phi(1, 1, 1) = -1$. The other result is immediate from the formula for a coboundary. \diamond

LEMMA 4 *Let ϕ be a cocycle defined in \mathbf{Z}_n . Then for all $x \in \mathbf{Z}_n$ we have,*

- 1) $\phi((n-1)x, x, (n-1)x)\phi(x, (n-1)x, x) = 1$.
- 2) $\phi((n-1)x, x, x) \cdot \phi(x, x, (n-1)x) = \frac{1}{\phi((n-1)x, 2x, (n-1)x)}$.

Proof Follows from the definition of a cocycle and $nx = 0$ for all x . \diamond

LEMMA 5 *Let ϕ be a cocycle defined in \mathbf{Z}_3 . Then $\forall x \in \mathbf{Z}_3$ we have,*

- 1) $\phi(2x, x, 2x)\phi(x, 2x, x) = 1$
- 2) $\phi(2x, x, x)\phi(x, x, 2x) = \frac{1}{\phi(2x, 2x, 2x)}$
- 3) $(\phi(x, x, x)\phi(2x, 2x, 2x))^3 = 1$
- 4) $\phi(2x, 2x, x)\phi(2x, x, 2x) = \phi(x, x, 2x)$
- 5) $\phi(x, x, x)\phi(x, x, 2x) = \frac{\phi(2x, x, 2x)}{\phi(x, 2x, 2x)}$.

Proof Parts 1) and 2) follow by Lemma 4. Part 3) is $\phi(x, x, x)^2 = \frac{\phi(2x, x, x)\phi(x, x, 2x)}{\phi(x, 2x, x)}$ but $\phi(2x, x, x)\phi(x, x, 2x) = \frac{1}{\phi(2x, 2x, 2x)}$. Then $\phi(x, x, x)^2 \cdot \phi(2x, 2x, 2x)\phi(x, 2x, x) = 1$ and analogously $\phi(2x, 2x, 2x)^2 \cdot \phi(x, x, x)\phi(2x, x, 2x) = 1$. So by 1) we have that $\phi(x, x, x)^3\phi(2x, 2x, 2x)^3 = 1$. Parts 4) and 5) follow directly by the definition of a cocycle. \diamond

PROPOSITION 6 *Every cocycle on \mathbf{Z}_3 has the form*

$$\begin{aligned} \phi_{111} &= \alpha, & \phi_{112} &= \beta, & \phi_{121} &= \frac{1}{\omega\alpha}, & \phi_{122} &= \frac{\omega}{\beta} \\ \phi_{211} &= \frac{\alpha}{\beta\omega}, & \phi_{212} &= \alpha\omega, & \phi_{221} &= \frac{\beta}{\omega\alpha}, & \phi_{222} &= \frac{\omega}{\alpha} \end{aligned}$$

for some non zero $\alpha, \beta \in k$ and ω a cubic root of the unity. Here $\phi(1, 1, 1) = \phi_{111}$, etc. is a shorthand.

Proof First of all, let $\omega = \phi(1, 1, 1)\phi(2, 2, 2)$, a cubic root of unity by part 3) of the last lemma. We also have $\phi(1, 1, 1)^2 = \frac{\phi(2, 1, 1)\phi(1, 1, 2)}{\phi(1, 2, 1)} = \frac{1}{\phi(1, 2, 1)\phi(2, 2, 2)}$ by part 2) of the last lemma. Hence $\phi(1, 1, 1) = \frac{1}{\omega\phi(1, 2, 1)}$. On the other hand $\phi(1, 1, 2)\phi(1, 2, 2) = \phi(2, 2, 2)\phi(1, 1, 1) = \omega$. Also from part 2) of the lemma, we have $\phi(2, 1, 1)\phi(1, 1, 2) = \frac{1}{\phi(2, 2, 2)}$ and from part 4) we have $\phi(2, 2, 1)\phi(2, 1, 2) = \phi(1, 1, 2)$. Similarly part 5) gives $\phi(1, 1, 1)\phi(1, 1, 2) = \frac{\phi(2, 1, 2)}{\phi(1, 2, 2)}$. Denoting $\phi(1, 1, 1) = \alpha$ and $\phi(1, 1, 2) = \beta$, we have the result as stated. \diamond

This can be written, for example, as

$$\phi(x, y, z) = (\alpha^{(-1)^z + x - xz} \beta^{x-z})^{(-1)^y} \begin{cases} 1 & \text{if } x = y = 1 \\ \omega^z & \text{else} \end{cases}$$

for $x, y, z \neq 0$.

PROPOSITION 7 *A cocycle on \mathbf{Z}_3 in the parametrisation above is coboundary iff $\omega = 1$.*

Proof Chose any cochain F (an invertible function such that $F(0, x) = F(x, 0) = 1$ for all x). For brevity we write its entries as a matrix $F(1, 2) = F_{12}$ etc. Let $\frac{F_{21}}{F_{12}} = \alpha$ and $\frac{F_{11}F_{22}}{F_{12}} = \beta$. Then a coboundary $\phi(x, y, z) = F(x, y)F(xy, z)/F(y, z)F(x, yz)$ is $\phi_{111} = \frac{F_{11}F_{21}}{F_{11}F_{12}} = \alpha$, $\phi_{112} = \frac{F_{11}F_{22}}{F_{12}} = \beta$, $\phi_{121} = \frac{F_{12}}{F_{21}} = \frac{1}{\alpha}$, $\phi_{122} = \frac{F_{12}}{F_{22}F_{11}} = \frac{1}{\beta}$, $\phi_{211} = \frac{F_{21}}{F_{11}F_{22}} = \frac{\alpha}{\beta}$, $\phi_{212} = \frac{F_{21}}{F_{12}} = \alpha$, $\phi_{221} = \frac{F_{22}F_{11}}{F_{21}} = \frac{\beta}{\alpha}$, $\phi_{222} = \frac{F_{22}F_{12}}{F_{22}F_{21}} = \frac{1}{\alpha}$ which is of the form above with $\omega = 1$. Conversely, if ϕ of the form above is a coboundary then $\phi_{111} = \frac{F_{11}F_{21}}{F_{11}F_{12}} = \alpha$, $\phi_{121} = \frac{F_{12}F_{01}}{F_{21}F_{10}} = \frac{1}{\omega\alpha}$. So $\omega = 1$. \diamond

PROPOSITION 8 *Every choice of invertible α, β, ω with $\omega^3 = 1$ yields a cocycle on \mathbf{Z}_3 of the form above. In particular,*

$$\phi(x, y, z) = \begin{cases} 1 & \text{if } x = y = 1 \\ \omega^z & \text{else} \end{cases}$$

for $x, y, z \neq 0$ is a noncoboundary cocycle when $\omega \neq 1$ and every cocycle is cohomologically equivalent to one of this form.

Proof We take $\alpha = \beta = 1$ and ω a nontrivial cube root of unity in Proposition 6 and verify directly that it is indeed a 3-cocycle. The cocycle condition is empty when any of the arguments is 0, so we assume that they are not. Then, as we have two different expressions for ϕ , we consider the cases (i) $\phi(1, 1, z)\phi(1, z, w) = \frac{\phi(2, z, w)\phi(1, 1, z+w)}{\phi(1, 1+z, w)}$ is satisfied because both sides are 1 if $z = 1$ and ω^w when $z = 2$. (ii) $\phi(x, 1, 1)\phi(1, 1, w) = \frac{\phi(x+1, 1, w)\phi(x, 1, 1+w)}{\phi(x, 2, w)}$ is satisfied because both sides are 1 if $x = 1$ and ω if $x = 2$. (iii) $\phi(x, 1-x, 1)\phi(1-x, 1, w) = \frac{\phi(1, 1, w)\phi(x, 1-x, 1+w)}{\phi(x, 2-x, w)}$ is satisfied because both sides are 1 if $x = 1$ and ω^{w+1} if $x = 2$. (iv) $\phi(1, y, 1-y)\phi(y, 1-y, w) = \frac{\phi(1+y, 1-y, w)\phi(1, y, 1-y+w)}{\phi(1, 1, w)}$ is satisfied because both sides are 1 if $y = 1$ and ω^{w-1} if $y = 2$. On the other hand, we know by Proposition 6 that every cocycle is the product of this one defined by some ω and one of the coboundary type defined by α, β in Proposition 7. \diamond

A more symmetric choice to generate the cohomology is with $\alpha = \beta = \omega^2$. This can be written more compactly as

$$\phi(x, y, z) = \omega^{xz - xy - yz}$$

for $x, y, z \neq 0$, and is cohomologically equivalent to the cocycle in Proposition 8.

COROLLARY 9 *A cocycle on \mathbf{Z}_3 is trivial if and only if there is an element $x \neq 0$ in \mathbf{Z}_3 such that $\phi(x, y, z) = 1$ for all y, z*

Proof If $\phi(1, y, z) = 1$ for all y, z , we have $\phi_{112} = \phi_{111} = \phi_{121} = 1$ and hence by Proposition 6 we have $\alpha = \beta = \omega = 1$ and $\phi = 1$. If $\phi(2, y, z) = 1$ for all y, z we have that $\phi_{212} = \phi_{211} = \phi_{222} = 1$ and hence $\alpha = \beta = \omega = 1$ again. \diamond

Returning to the general case, a natural cocycle motivated by some of the above is:

COROLLARY 10 *Let q be an n -th root of unity. Then*

$$\phi(x, y, z) = q^{xyz}$$

is a cocycle on \mathbf{Z}_n . When $n = 3$ it is a coboundary with $\alpha = q, \beta = q^2, \omega = 1$.

Proof The 3-cocycle condition becomes $yzw + x(y+z)w + xyz = xy(z+w) + (x+y)zw$ in \mathbf{Z}_n and holds using distributivity of the product in the ring \mathbf{Z}_n over the additive group structure. When $n = 3$ it must fit into our classification above, which it does with $\omega = 1, \alpha = q, \beta = q^2$. \diamond

Note that if q is not a root of unity, we still have a coboundary,

$$\phi_{111} = \phi_{212} = \phi_{221} = q, \quad \phi_{121} = \phi_{211} = \phi_{222} = q^{-1}, \quad \phi_{122} = q^{-2}, \quad \phi_{112} = q^2$$

if we choose $\alpha = q, \beta = q^2$ again.

Let us stress that every cocycle leads to a category of quasialgebras and that these are different even if the cocycles are cohomologically equivalent, i.e we are interested in the full parametrisation in Proposition 6. When related by a coboundary the quasialgebras may potentially be related to each other by twisting in the same way as the octonions are a twist of the group algebra of $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2[1]$. When in different cohomology classes then the quasialgebras cannot be related by a twist and are in this sense ‘topologically distinct’ examples.

EXAMPLE 11 *Let q be a cubic root of unity. The \mathbf{Z}_3 -graded quasialgebra with graded basis $\{e_x\}$ for $x \in \mathbf{Z}_3$, $e_0 = 1$ and other products*

$$e_x e_y = e_{x+y} q^{y-x}, \quad \forall x, y \neq 0$$

is a twisting of $k\mathbf{Z}_3$ and has coboundary cocycle $\phi(x, y, z) = q^{xyz}$.

Proof We take $F(x, y) = q^{y-x}$ for $x, y \neq 0$ in Proposition 7. This has $\alpha = q, \beta = q^2$ and hence $\partial F = \phi$ in Corollary 10. \diamond

On the other hand, Theorem 7.3 of [1] provides a construction of a quasialgebra for *any* cocycle (and any graded vector space) as the quasialgebra of quasi-matrices. In particular, let ϕ be a cocycle on \mathbf{Z}_n then the natural quasialgebra of quasimatrices $M_{n,\phi}$ has basis E_{ij} labelled by $i, j \in \mathbf{Z}_n$ and of degree $i - j$, with the product

$$E_{ij} \cdot E_{kl} = \delta_{jk} E_{il} \frac{\phi(i, -j, j-l)}{\phi(-j, j, -l)}. \quad (1)$$

The quasiassociativity is

$$(E_{ij} \cdot E_{kl}) \cdot E_{rs} = E_{ij} \cdot (E_{kl} \cdot E_{rs}) \phi(i-j, k-l, r-s) \quad (2)$$

which can be computed more explicitly depending on the form of ϕ .

EXAMPLE 12 Let $q^n = 1$. Then $M_{n,\phi}$ for ϕ in Corollary 10 has the product

$$E_{ij} \cdot E_{kl} = \delta_{jk} E_{il} q^{ijl-j^2(i+l)}.$$

Let $\omega^3 = 1$. Then $M_{3,\phi}$ for the noncoboundary ϕ in Proposition 8 has the product

$$E_{ij} \cdot E_{kl} = \delta_{jk} E_{il} \begin{cases} \omega^l & \text{if } i = 0, j \neq 0 \text{ or } i = 1, j = 2 \\ \omega^j & \text{else.} \end{cases}$$

Proof We insert the form of the relevant cocycle into (1). In the second case all the possibilities for $i, j, l, j-l$ zero or not have to be looked at separately but can afterwards be recombined as stated. \diamond

Here $M_{2,\phi}$ for $q \neq 1$ is also noncoboundary and an example of the second type in Corollary 3. Also, the corresponding quasimatrix product[1] among actual matrices a, b has the same form

$$(a \cdot b)_{il} = \sum_j a_{ij} b_{jl} \frac{\phi(i, -j, j-l)}{\phi(-j, j, -l)} \quad (3)$$

and therefore the same coefficients for the $M_{n,\phi}$, $M_{3,\phi}$ as appearing in Example 12. For example, $M_{2,\phi}$ has the product

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \cdot \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} = \begin{pmatrix} a_{00}b_{00} + a_{01}b_{10} & a_{00}b_{01} - a_{01}b_{11} \\ a_{10}b_{00} - a_{11}b_{10} & a_{10}b_{01} - a_{11}b_{11} \end{pmatrix}.$$

Finally, we note that the cocycle in Corollary 10 has an obvious generalisation to $(\mathbf{Z}_n)^m$ as

$$\phi(\vec{x}, \vec{y}, \vec{z}) = q^{(\vec{x}, \vec{y}, \vec{z})}$$

where we use a vector notation with components in \mathbf{Z}_n and $(\ , \ , \)$ is \mathbf{Z}_n -trilinear. The cocycle for the octonions is a coboundary example of this type on $(\mathbf{Z}_2)^3$.

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